Finite Field Kakeya with and without polynomials

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"How small can a set in the plane be in which you can turn a unit needle completely around?" This was the question which gave birth to the yet unsolved Kakeya problem more then 100 years ago. In 1999 Wolff proposed a new version of the problem [Theorem 1] in a finite vector space instead of $\mathbb{R}^n$, which was then thought to be almost as hard as the original problem. In 2008 Dvir published a marvelous proof that uses only two simple lemmas [3 and 5] and can even fit in very narrow margin!

Notations and definitons:

- $\mathbb{F}$ will always denote a field and $\mathbb{F}_q^n$ is the n-dimensional vector space over the finite field with q elements.
- We denote by $\text{Poly}_D(\mathbb{F}^n)$ the space of polynomials in n variables, with coefficients in $\mathbb{F}$ and of degree at most D.
- A Kakeya set $K \subset \mathbb{F}_q^n$ is a set that contains a line in every direction. More formally, K is a Kakeya set if for all $a \in \mathbb{F}_q^n \setminus \{0\}$ there exists some $b \in \mathbb{F}_q^n$ such that $\{at + b : t \in \mathbb{F}_q\} \subset K$.

1 Introduction to the polynomial method

We begin this first part with our main theorem, the most celebrated example of the polynomial method.

Theorem 1 (Dvir)**.** *For a Kakeya set $K \subset \mathbb{F}_n^q$ it holds*

$$|K| \geq c_n q^n$$

for $c_n = (10n)^{-n}$.

Proof idea. To prove the statement we assume the contrary, that $K \subset \mathbb{F}_n^q$ is small. Using the *parameter counting argument* we can find a non-zero polynomial P of degree $D < q$ vanishing on K. But knowing that K is a Kakeya set, using the *vanishing lemma*, we imply that P vanishes on lines with <u>q points</u> pointing in <u>all</u> directions. From this fact it follows that the coefficients near the highest degree terms of P vanish as well, contradicting the degree of the polynomial P. □

Lemma 2 (Parameter counting)**.** *Let $S \subset \mathbb{F}^n$ be a finite set. If $\dim(Poly_D(\mathbb{F})) > |S|$, then there exists a non-zero polynomial in $Poly_D(\mathbb{F})$ that vanishes on S.*

Proof. Consequence of Rank-nullity theorem applied on evaluation map $\text{Poly}_D(\mathbb{F}) \to \mathbb{F}^{|S|}$. □

Corollary 3. *For $n \geq 2$, for a finite subset $S \subset \mathbb{F}^n$, there is a non-zero polynomial of degree $\leq n|S|^{1/n}$ that vanishes on S.*

Lemma 4. *If $P \in Poly_D(\mathbb{F})$ vanishes at $D+1$ points, it is the zero polynomial.*

Proof. Consequence of division with remainder in the ring of polynomials. □

Corollary 5 (Vanishing lemma)**.** *Let $l \subset \mathbb{F}^n$ be a line, i.e. a 1-dim affine subspace. If $P \in Poly_D(\mathbb{F})$ vanishes at $D+1$ points of l, then P vanishes on the whole l.*

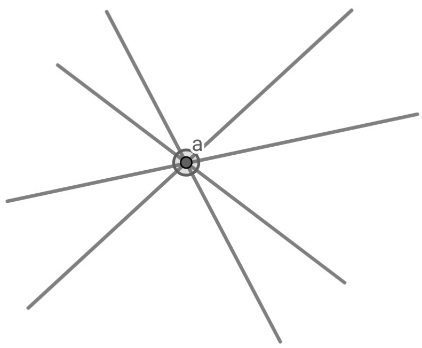
2 The story before the polynomials

All the methods that do not use polynomials, at least those presented here, have one thing in common, they do not exploit all the information contained in the definiton of a Kakeya set.

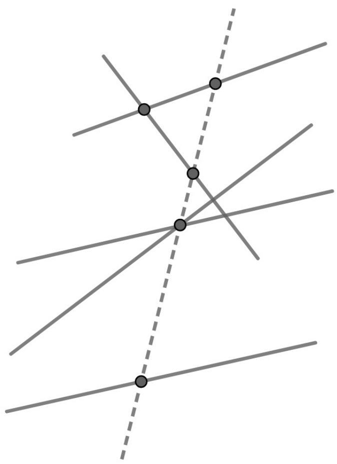
Proposition 6 (The naive method)**.** *Suppose* $s \leq q$*. If* $l_1, l_2, ..., l_s$ *are lines in* $\mathbb{F}_q^n$*, their union has cardinality* $\geq \frac{1}{2}qs$*.*
In particular, for a Kakeya set $K \subset \mathbb{F}_q^n$ *it holds* $|K| \geq \frac{1}{2}q^2$*.*

Proposition 7 (Bush method)**.** *If* $l_1, l_2, ..., l_s$ *are lines in* $\mathbb{F}_q^n$*, their union has cardinality* $\geq \frac{1}{2}qs^{1/2}$*.*
In particular, for a Kakeya set $K \subset \mathbb{F}_q^n$ *it holds* $|K| \geq \frac{1}{2}q^{\frac{n+1}{2}}$*.*

Proposition 8 (Hairbrush method [Wolff])**.** *If* $l_1, l_2, ..., l_s$ *are lines in* $\mathbb{F}_q^n$ *and only at most* $q+1$ *of them lie in any 2-dimensional plane, then their union has cardinality* $\geq \frac{1}{2}q^{3/2}s^{1/2}$*.*
In particular, for a Kakeya set $K \subset \mathbb{F}_q^n$ *it holds* $|K| \geq \frac{1}{2}q^{\frac{n+2}{2}}$*.*



(a) Bush with stem a



(b) Hairbrush with dotted-line stem